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Wigner-Dyson statistics for a class of integrable models

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We construct an ensemble of second-quantized Hamiltonians with two bosonic degrees of freedom, whose members display with probability one Gaussian orthogonal ensemble (GOE) or Gaussian unitary ensemble (GUE) statistics. Nevertheless, these Hamiltonians have a second integral of motion, namely, the boson number, and thus are integrable. To construct this ensemble we use some ''reverse engineering'' starting from the fact that n bosons in a two-level system with random interactions have an integrable classical limit by the old Heisenberg association of boson operators to actions and angles. By choosing an n-body random interaction and degenerate levels we end up with GOE or GUE Hamiltonians. Ergodicity of these ensembles completes the example.

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Recently, there has been considerable interest in spinless n-boson systems with k-body random interactions [1,2], both for degenerate [3] and nondegenerate [4,5] single-particle levels. In particular, anomalous statistics for the two-level system have been understood from the fact that these systems are integrable [6]. The two-level ensemble corresponds to a time-independent two degrees of freedom system, in the sense that creation and annihilation operators for the two single-particle levels are canonical operators for the system. The second integral of motion corresponds to the conservation of the number of bosons. The Hamiltonian is a function of the creation and annihilation operators, whose precise form depends on the question whether we have two-body interactions or more complicated many-body interactions. The number of (random) coefficients in this model increases quadratically with the rank k of the interaction. A classical Hamiltonian can formally be written for any number of particles but, in fact, only for large particle number the quantum problem reaches the classical limit [7,8]. Thus the boson number plays the role of an action. For fixed particle number, the model can be reduced to a Hamiltonian of one degree of freedom with the number of bosons appearing as a parameter.

In this paper we shall consider a different large-n limit for the two-level system. Our model consists in choosing the rank of the interaction equal to the particle number, $k=n$. Hence the Hamiltonian changes for different values of the particle number. This choice, by definition of the ensemble, directly leads to a Gaussian orthogonal ensemble (GOE) or Gaussian unitary ensemble (GUE) matrix [9]. We then consider the limit of large particle number. Clearly, we will have an ensemble of systems which have GOE or GUE statistics with probability 1 [10], while being integrable in a well-defined limit of large actions. This represents an important caveat concerning the idea that Wigner-Dyson statistics are characteristic of classical chaotic systems. Note that this has no bearing on the fairly well established quantum chaos conjecture [11–17], which establishes that classical chaos implies typically Wigner-Dyson statistics. In a similar vein, Wu *et al.* [18] have performed the following calculation: after having chosen an unfolded *fixed* spectrum of given length N generated via the diagonalization of a matrix taken from the GOE, they fit a one-dimensional potential, the spectrum of which coincides with the above spectrum for its first N values. The main difference between their work and ours is that we display explicitly an ensemble of integrable Hamiltonians having a GOE spectrum.

The Hamiltonian in second quantization is

$$\hat{H}=\sum_{s,t=0}^{k}\frac{v_{s,t}}{\mathcal{N}_{s,t}}(b_1^\dagger)^s(b_2^\dagger)^{k-s}(b_1)^t(b_2)^{k-t}. \tag{1}$$

Here, $b_j^\dagger$ creates a spinless boson in single-particle level j ($j=1,2$), b_j destroys it, and the k-body matrix elements $v_{s,t}$ correspond to a GOE or a GUE [3]. The number operator $\hat{n}=b_1^\dagger b_1+b_2^\dagger b_2$ commutes with the Hamiltonian independently of the rank k of the interaction. The combinatorial factors $\mathcal{N}_{s,t}=[s!(k-s)!t!(k-t)!]^{1/2}$ in Eq. (1) are introduced in order to have *exactly* a GOE or GUE Hamiltonian for $n=k$, where n is the number of bosons [1,2]. For the two-level system, the dimension of Hilbert space is $N=n+1$.

The classical Hamiltonian is obtained as follows [6,19]: $\hat{H}$ is symmetrized with respect to the ordering of the creation and annihilation operators. This permits a correct assignment of the zero-point energy and therefore a one-to-one classical-quantum energy comparison. Then, we use Heisenberg semiclassical rules [7]

$$b_j^\dagger\to I_j^{1/2}\exp(i\phi_j),\quad b_j\to I_j^{1/2}\exp(-i\phi_j). \tag{2}$$

Formally, the classical Hamiltonian can be written as $H=H_0+V$. The unperturbed Hamiltonian H_0, which includes a constant associated with the zero-point energy correction, depends only upon the actions I_1 and I_2; the residual interaction V depends also on the angles ϕ_1 and ϕ_2. These terms are explicitly given by

$$H_0=\sum_{s=0}^{k}\frac{v_{s,s}}{\mathcal{N}_{s,s}}P_s\left(I_1-\frac{1}{2},s\right)P_{k-s}\left(I_2-\frac{1}{2},k-s\right), \quad (3)$$

$$V=\sum_{s>t}\frac{v_{s,t}(I_1I_2)^{(s-t)/2}}{2\mathcal{N}_{s,t}}\cos[(s-t)(\phi_1-\phi_2)]\left[P_t\left(I_1-\frac{1}{2},s\right)+P_t\left(I_1-\frac{1}{2},t\right)\right]\left[P_{k-s}\left(I_2-\frac{1}{2},k-s\right)+P_{k-s}\left(I_2-\frac{1}{2},k-t\right)\right]. \quad (4)$$

In these equations (details of their derivation will be given elsewhere), $P_t(I,s)$ is a polynomial of degree t on I defined as $(s\geqslant t\geqslant 0)$,

$$P_t(I,s)=\prod_{i=1}^{t}[I-(s-i)]. \quad (5)$$

Here we have assumed real matrix elements $v_{s,t}$, which are independent random variables, Gaussian distributed with zero mean and variance given by $\overline{v_{s,t}v_{s',t'}}=\delta_{ss'}\delta_{tt'}+\delta_{st'}\delta_{ts'}$, with the overline denoting ensemble average. Hence the k-body interaction matrices form a GOE. The analogous case for the GUE is obtained by replacing in Eq. (4) $v_{s,t}$ by $|v_{s,t}|$ and introducing a phase $\omega_{s,t}$ in the cosine function.

The Hamiltonian is written as a polynomial of the two actions with random coefficients, and cosines of the difference of the two angles. The invariant is given by $K=I_1+I_2=n+1$ which reflects the translational symmetry of the interaction with respect to the angles. Again we can test the corresponding Poisson brackets [6]. This form displays explicitly that we deal with an integrable two degrees of freedom problem. If we fix the invariant, i.e., the particle number n, we can reduce the Hamiltonian to one degree of freedom. The quantum Hamiltonian acts on a Hilbert space of dimension $n+1$, which is the number of ways we can distribute the n bosons in the two single-particle levels. Formally, this corresponds to the so-called ''polyads'' for algebraic Hamiltonians in molecular systems [19]. The reduced Hamiltonian is obtained by performing the canonical transformation defined by the generating function $W=K\phi_1+J(\phi_2-\phi_1)$. The new actions K and J and their corresponding canonically conjugated angles χ and ψ are related to the old variables by

$$I_1=K-J, \quad I_2=J, \quad (6a)$$

$$\chi=\phi_1, \quad \psi=\phi_2-\phi_1. \quad (6b)$$

Substituting the new variables in Eqs. (3) and (4), and fixing K yields explicitly the reduced Hamiltonian.

We now have established the main point of the paper: if we consider $k=n$, by construction the quantum Hamiltonian (1) coincides with a GOE [1,3]. Each member of the ensemble is associated, in the limit of large $n(=k)$, to a classical Hamiltonian with one effective degree of freedom. This defines the ensemble of integrable Hamiltonians.

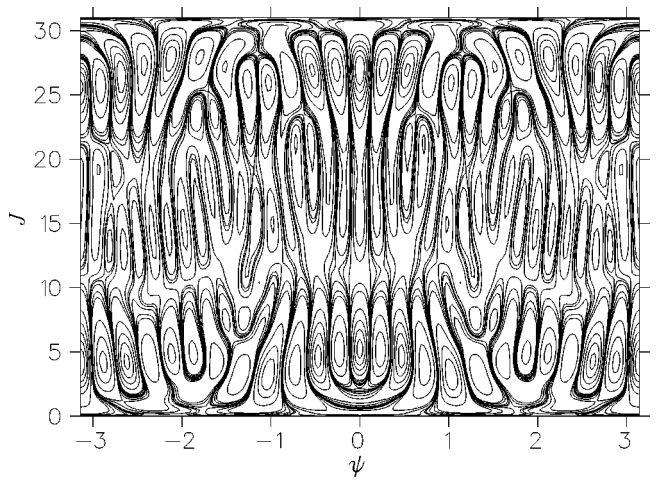

FIG. 1. Phase-space structure of a specific realization of the reduced Hamiltonian for $k=n=30$.

Let us now qualitatively understand why this happens. For this purpose we investigate the structure of the phase space of a typical member of the ensemble for fixed $n\gg 1$. A Poincaré surface of the two degrees of freedom system constructed for constant $K=n+1$ is equivalent to a contour plot of the reduced one-dimensional Hamiltonian. Fixing K makes for a compact phase space in the reduced coordinates ψ and J defined above. The volume corresponds to $n+1$ as we normalized to one quantum state per unit cell. The number of random coefficients entering in the Hamiltonian grows quadratically with k. The equations of motion then contain polynomials in J and $K-J$ (of degree up to $k=n$) multiplied by sines or cosines of multiples of ψ. In this case, the degree of the polynomials sets $\sim n^2$ as the upper limit for the total number of fixed points (stable and unstable). While a more precise estimate for their number is difficult to establish, our numerical results show that the fixed points proliferate more rapidly than n, and come close to the upper bound. The typical phase-space portrait will therefore consist of elliptic islands surrounded by separatrices in a complicated mesh. In Fig. 1 we illustrate the phase-space structure of a member of the ensemble for $k=n=30$. The tori whose action satisfies the Einstein-Brillouin-Keller (EBK) quantization condition

$$S(E_i)\equiv\frac{1}{2\pi}\oint J(E_i)d\psi=\frac{1}{2\pi}\left(n_i+\frac{\alpha_i}{4}\right), \quad (7)$$

define implicitly the ith energy level E_i. Here, n_i is an integer and α_i is the Maslov index of the orbit. Note that there is *a priori* no monotonic dependence of the value of the action as a function of energy for the reduced Hamiltonian. That is, for a given energy two or more distinct invariant tori may exist in different regions of phase space; each of these tori may have a different value of the corresponding action.

As mentioned above, the number of stable and unstable fixed points proliferates very rapidly for growing $k=n$. This is a consequence of the many-body character of the interaction V in Eq. (4). In particular, in the limit we consider, this

number grows much faster than $n \sim \hbar^{-1}$; this implies that the phase-space volume surrounding each stable fixed point shrinks for $n=k \to \infty$.

For one degree of freedom systems in the large-n limit with k fixed the spectrum is constructed from sequences of levels obtained by torus quantization around the elliptic fixed points, i.e., most tori which satisfy the EBK condition can be uniquely assigned to one elliptic fixed point. The spectrum is thus a superposition of picket-fence (equidistant) sequences of levels as reported in Ref. [3] for small values of k. For fixed (sufficiently large) values of k we expect a Poisson spectrum to arise on short length scales. This expectation is based on the fact that WKB theory yields a superposition of many picket-fence (equidistant) spectra with different spacings.

In contrast, in the present case ($k=n$) we have $\sim n^2$ elliptic fixed points and only $n+1$ energy levels. Therefore, a stable periodic orbit that fulfills the EBK condition (7), instead of surrounding an elliptic point, has to accommodate and explore more extended regions in phase space, coming close to the separatrix associated with many different unstable fixed points. Stated in a different way, while the periodic orbits of the Hamiltonian are strictly stable, wave packets started on initial conditions separated by a distance of the order $\hbar^{1/2} \sim n^{-1/2}$ will sample very different regions of phase space. This coarse graining will effectively mimic unstable motion. This fact makes plausible that we do not obtain a smooth spectrum that can be unfolded to yield (superpositions of) picket-fence spectra. Indeed, we find random matrix spectral fluctuations. Note that the results on the nonergodic behavior in the dense limit (k fixed) of the bosonic k-body embedded ensembles [3] cease to apply in this case by the same line of reasoning, and therefore we recover the standard ergodicity of the GOE [10].

Finally, we consider the extension to the GUE case. We recall that in action and angle variables of the harmonic oscillator, time-reversal invariance is tested by the invariance of the Hamiltonian under the transformation $\phi_j \to -\phi_j$ (for $j=1,2$). As mentioned above, when considering complex k-body matrix elements, Eq. (4) requires certain modifications; namely, $v_{s,t}$ is replaced by its modulus $|v_{s,t}|$ and the phases $\omega_{s,t}$ are introduced in the argument of the cosine functions. The introduction of the phases $\omega_{s,t}$ in the cosine functions of Eq. (4) breaks the time-reversal invariance.

Summarizing, we have defined an ensemble of integrable Hamiltonians of two degrees of freedom, or equivalently, an ensemble of one degree of freedom Hamiltonians by fixing the number of bosons n. In the limit $n=k \to \infty$, each member of the (reduced) ensemble is mapped exactly by the quantization onto a member of the GOE or GUE. Therefore, the fluctuation properties of its eigenvalues follow the predictions of random matrix theory. This result can be interpreted in two ways. First, the limiting Hamiltonian is considered to be a classical one, or second, such a limit is not accepted as a classical one. In either case, for this family of Hamiltonians Wigner-Dyson fluctuations do not imply chaos in classical dynamics.

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[1] V.K.B. Kota, Phys. Rep. **347**, 223 (2001).
[2] L. Benet and H.A. Weidenmüller, J. Phys. A **36**, 3569 (2003).
[3] T. Asaga *et al.*, Europhys. Lett. **56**, 340 (2001); Ann. Phys. (N.Y.) **298**, 229 (2002).
[4] K. Patel, M.S. Desai, V. Potbhare, and V.K.B. Kota, Phys. Lett. A **275**, 329 (2000).
[5] N.D. Chavda, V. Potbhare, and V.K.B. Kota, Phys. Lett. A **311**, 331 (2003).
[6] L. Benet, C. Jung, and F. Leyvraz, J. Phys. A **36**, L217 (2003).
[7] W. Heisenberg, Z. Phys. **33**, 879 (1925).
[8] L.G. Yaffe, Rev. Mod. Phys. **54**, 407 (1982).
[9] M.L. Mehta, *Random Matrices* (Academic Press, New York, 1991); T. Guhr, A. Müller-Groehling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[10] J.B. French, P.A. Mello, and A. Pandey, Phys. Lett. **80B**, 17 (1978).
[11] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[12] S.W. McDonald and A.N. Kaufmann, Phys. Rev. Lett. **42**, 1189 (1979).
[13] G. Casati, F. Valz-Gris, and I. Guarneri, Lett. Nuovo Cimento **28**, 279 (1980).
[14] M.V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981); in *The Wave-Particle Dualism*, edited by S. Diner *et al.* (Reidel, Dordrecht, 1984), p. 231.
[15] M.V. Berry, Proc. R. Soc. London, Ser. A **400**, 229 (1985).
[16] A.V. Andreev *et al.*, Phys. Rev. Lett. **76**, 3947 (1995).
[17] F. Leyvraz and T.H. Seligman, in *Proceedings of the Fourth Wigner Symposium*, edited by N. Atakishiev, T.H. Seligman, and K.B. Wolf (World Scientific, Singapore, 1996), p. 429.
[18] H. Wu, M. Vallières, D.H. Feng, and D.W.L. Sprung, Phys. Rev. A **42**, 1027 (1990).
[19] M.P. Jacobson, C. Jung, H.S. Taylor, and R.W. Field, J. Chem. Phys. **111**, 600 (1999).